### COMP3121/9101 Week02 班课

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## Asymptotic notation

- "Big Oh"

- f(n) = O(g(n)) is an abbreviation for: “There exist \*\*positive\*\* constants c and n0 such that 0≤f(n)≤cg(n) for all \*\*n≥n0\*\*”.

- 上限

- "Omega"

- f(n) = Ω(g(n)) is an abbreviation for: “There exists \*\*positive\*\* constants c and n0 such that 0≤cg(n)≤f(n) for all \*\*n≥n0\*\*.”

- g(n)=O(f(n))

- 下限

- "Theta"

- f(n) = Θ(g(n)) if and only if \*\*f(n) = O(g(n))\*\* and \*\*f(n) = Ω(g(n))\*\*; thus, f(n) and g(n) have the same asymptotic growth rate.

- 相等

## Recurrences

Let a ≥ 1 be an integer and b > 1 a real number; Assume that a \*\*divide-and-conquer algorithm\*\*:

reduces a problem of size n to a many problems of smaller size \*\*n/b\*\*; the overhead cost of splitting up/combining the solutions for size n/b into a solution for size n is if \*\*f(n)\*\*,

then the time complexity of such algorithm satisfies

$$

T(n)=aT(ceiling(\frac{n}{b}))+f(n)\\

T(n)=aT(\frac{n}{b})+f(n)

$$

- 只有少数recurrence会比较容易解，多数很复杂

- 我们并不需要准确的解，我们只需要一个大概的时间复杂度以体现算法的效率

## Master Theorem

Let:

- a ≥ 1 be an integer and and b > 1 a real;

- f(n) > 0 be a non-decreasing function;

- T (n) be the solution of the recurrence T (n) = a T (n/b) + f (n);

Then:

1. If $f(n) = O(n^{\log\_b a−ε})$ for some ε > 0, then $T(n) = Θ(n^{log\_b a})$;

2. If $f(n) = Θ(n^{\log\_b a})$, then $T(n) = Θ(n^{\log\_b a} \log\_2 n)$;

3. If $f(n)=Ω(n^{\log\_ba+ε})$ for some $ε>0$, and for some \*\*c<1\*\* and some $n\_0$,

$af (n/b) ≤ cf(n)$ holds for all $n > n\_0$, then $T(n) = Θ(f(n))$;

4. If none of these conditions hold, the Master Theorem is NOT applicable.

### Examples

(Queenie goes through lecture examples)

$T(n) = 3T(\frac{n}{2}) + n(2 + \cos n)$

a=3

b = 2

f(n)= $n(2 + \cos n)$

$n^{\log\_ba}=n^{\log\_23}\approx n^{1.58}>n$

2+cos n [1, 3]

f(n)=$O(n^{1.58-\epsilon})$ for $\epsilon$ <0.5

T(n)=$\Theta(n^{\log\_23})$

\*\*Solution \*\*

For the given equation $T(n) = 3T(\frac{n}{2}) + n(2 + cos n)$ ,

$a = 3$, $b = 2$, $f(n) = n(2 + cos n) = O(n)$ because $cosn$ is bounded at $[-1, 1]$, $2 + \cos n$ is a positive constant.

so we have $a \ge 1, b > 1$, $n^{log\_ba} = n^{log\_23}\approx n^{1.58}$, then $f(n) = O(n^{log\_23-\epsilon})$, for some $0<\epsilon < 0.4$, which is case 1 of Master Theorem, then $T(n) = \Theta(n^{log\_23})$.

$T(n) = 3T(\frac{n}{4}) + n^{\frac{4}{3}}$

a = 3, b = 4

n^log\_b a = n^{log\_4 3}\approx n^{0.79}<n^1 = n

f(n)=n^{4/3}

\Omega (f(n)=n^{log\_4 3 + \epsilon}) epsilon < 0.3

3f (n/4) ≤ cf(n)

$3(n/4)^{4/3}\le cn^{4/3}$

c > 3/4^{4/3}=0.47...

\*\*Solution\*\*

For the given equation $T(n) = 3T(\frac{n}{4}) + n^{\frac{4}{3}}$,

$a = 3$, $b = $4, $f(n) = n^{\frac{4}{3}}$ ,

so we have $a \ge 1, b > 1$, $n^{log\_ba} = n^{log\_43}\approx n^{0.78}$, then $f(n) = \Omega(n^{log\_34+\epsilon})$, for some $0<\epsilon < 0.07$, there exists some constant $c > 0$ such that $af(n/b) = cf(n)$, which is the third case of Master Theorem, then $T(n) = \Theta(f(n)) = \Theta(n^{\frac{4}{3}})$.

$T(n) = 5T(\frac{n}{2}) + n^{log\_2 5}(1 + sin(\frac{2\pi n}{3}))$

\*\*Solution\*\*

For the given equation $T(n) = 5T(\frac{n}{2}) + n^{log\_2 5}(1 + sin(\frac{2\pi n}{3}))$,

$a = 5$, $b = 2$, $f(n) =n^{log\_2 5}(1 + sin(\frac{2\pi n}{3}))$,

so we have $a \ge 1, b > 1$, let $g(n) = n^{log\_ba} = n^{log\_25}$, then we should have $f(n) = \Theta(g(n))$to apply master theorem.

To show, $f(n) = \Theta (g(n))$ there exist constant $c\_1, c\_2$ and $n\_0$ such that

$0 \leq c\_1g(n) \leq f(n) \leq c\_2 g(n)$ for all $n \geq n\_0$.

We know that $n$ is an integer, then $sin(\frac{2\pi n}{3}))$ is bounded at $[-\frac{\sqrt3}{2}, \frac{\sqrt3}{2}]$, hence $(1 + sin(\frac{2\pi n}{3})) $ is bounded at $[1-\frac{\sqrt3}{2}, 1+ \frac{\sqrt3}{2}]$. Thus we can take $0 < c\_1 \leq 1-\frac{\sqrt3}{2}$ and $c\_2 \geq 1+\frac{\sqrt3}{2}$, the equation holds for all $n > n\_0$. By master theorem, $T(n) =\Theta(n^{log\_2 5} lgn)$.

\*\*Solution\*\*

For the given equation $T(n) = T(n − 1) + log (n)$,

$a = 1$, $b = 1$, $f(n) = log(n)$, which does not satisfy the condition of Master Theorem.

$T(n) = T(n - 1) + log(n)\\

= T(n - 2) + log(n - 1) + log(n)\\

= T(n - 3) + log(n - 2) + log(n - 1) + log(n)\\

= T(0) + log(1) + ... + log(n)\\= T(0) + log(1\times2\times3\times...\times n)\\=T(0) + log(n!) \\\leq T(0) + log(n^n)\\=O(nlogn) $

## Karatsuba trick analysis

$T(n)=3(\frac{T}{2})􏰂 􏰃+cn$

$b=2; f(n)=cn; n^{\log\_ba} =n^{\log\_23} $

a=3;

since 1.5 < $\log\_2 3$ < 1.6 we have

$f(n)=cn=O(n^{\log\_23−ε})$ for any $0<ε<0.5$

Thus, the first case of the Master Theorem applies. Consequently,

$T (n) = Θ(n^{\log\_2 3}) < Θ(n^{1.585})$

### Dividing into 3 pieces

（lecture slides)

$T(n)=5T(\frac{n}{3})+cn$

### Slice into p+1 many slices

(b) T (n) = 2T (n/2) + √n + log n

a=2

b=2

n^{log\_22}=n^1

f(n)=n^(1/2)+log(n)

$f(n)= O(n^{1-\epsilon}) \epsilon <0.3$

Master theorem case 1